Review for Midterm II[1]

Assigned: March 30, 2021

Multivariable Calculus MATH 53
with Professor Stankova

Contents

1 Definitions

Be able to **write** precise definitions for any of the following concepts (where appropriate: both in words and in symbols), to **give** examples of each definition, and to **prove** that these definitions are satisfied in specific examples. Wherever appropriate, be able to **graph** examples for each definition. What is/are:

1. a *function of several variables*? The *domain* and *range* of such functions?
2. the *graph* of a two-variable function?
3. a *level curve* and a *contour map* of a two-variable function?
   a *level surface* of a three-variable function?
4. the *limit* (at a point) of a multi-variable function?
5. a *continuous* function of several variables? A *discontinuous* function?
   How to determine that a function is discontinuous?
6. the *operations* (addition, subtraction, multiplication, division, composition) on continuituous functions? When is continuity preserved?
7. a *partial derivative*? *Higher order* partial derivatives?
8. a *partial differential equation (PDE)*? The *Laplace equation*? What are some solutions?
9. a *differentiable function*? The *linear approximation* of a differentiable function? A *tangent plane* to a two-variable differentiable function?
10. a *tree diagram* of a function? A *branch* of such a tree? What are they used for?

11. *implicit differentiation*? When do we have to use it?
12. the *Wave equation*? *Traveling wave* to the right/left? What are some/all solutions?
13. *directional derivatives*? How are they related to partial derivatives?
14. *the gradient vector of a function*? What is its relation to maximal and minimal rates of change? What is its geometric interpretation?
15. the *tangent line* to the level curve of a two-variable function and the *tangent plane* to the level surface of a three-variable function? a *normal* to the level curve at point P?
16. a *critical point*? A *local* maximum/minimum? A *saddle point*?
17. a *global maximum/minimum*?
18. *interior*, *exterior*, and *boundary* points of a set?
19. a *closed set*? A *bounded set*? Why are closed bounded sets "nice" for optimization problems?
20. a *constrained* optimization problem?
21. a *Lagrange multiplier*?
22. the *AM-GM inequality*?
23. the *double integral* of a function of two variables over a rectangle? Over any region in the plane? What is its geometric interpretation?
24. a *Riemann sum* for a double integral over a rectangle?
25. the *average value* of a function over a 2d region?
26. *partial* integration?
27. *iterated* integral? How many such iterated integrals are there?
28. *cross-section* of a solid? *Cavalieri's principle*?
29. a *type I* region in the plane? A *type II*? What are they used for?
30. *switching* the *order* of integration?

2 Theorems

Be able to **write** what each of the following theorems (laws, propositions, corollaries, etc.) says. Be sure to understand, distinguish and **state** the conditions (hypothesis) of each theorem and its conclusion. Be prepared to **give** examples for each theorem, and most importantly, to **apply** each theorem appropriately in problems. The latter means: decide which theorem to use, check (in writing!) that all conditions of your theorem are satisfied in the problem in question, and then state (in writing!) the conclusion of the theorem using the specifics of your problem.

1. **"Contrapositive theorem"**: If $f(x,y)$ has different limits along two different paths approaching (a,b) (or one of them does not exist), then the limit of f at (a,b) does not exist.
2. **Continuity theorem:** Functions defined by algebraic expressions involving addition, multiplication, division, exponentiation, logs and (inverse) trig functions, and composition of such, are continuous where they are *well-defined* (i.e., where denominators are not zero, expressions inside square roots are non-negative, etc.).
3. **Clairaut's theorem:** If $f(x,y)$ has **continuous** mixed partial derivatives f_{xy} and f_{yx} on a disc $D_{(a,b)}$ inside the domain D_f, then $f_{xy} = f_{yx}$ on this disc.
4. **Sufficient condition for differentiability** (using partial derivatives): If $f_x(x,y)$ and $f_y(x,y)$ exist near (a,b) and are **continuous** at (a,b), then $f(x,y)$ is differentiable at (a,b).
5. The **linearization** of $f(x,y)$ at (x_0,y_0) is given by:
$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0).$$
The **tangent plane** to the graph of $f(x,y)$ at $(x_0,y_0,f(x_0,y_0))$ is given by:
$$z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0).$$
The **linear approximation** of $f(x,y)$ at $(x_0,y_0,f(x_0,y_0))$ is given by:
$$f(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0).$$

6. **The Chain Rule** for a variety of situations, paraphrased with the gradient vector.
7. **Implicit function theorem:** Suppose that x and y satisfy some equation $F(x,y)=0$. If $\frac{\partial F}{\partial y}(a,b)\neq 0$, then, *in theory*, the equation can be solved for value of x close to a, giving values of y close to b. *In practice*, however, it might not be possible to find a expression for $y(x)$; we can still find $\frac{\partial y}{\partial x}=-\frac{F_x}{F_y}$, using implicit differentiation.

   Similarly, if $F(x,y,z)=0$ and F_x, F_y, F_z exist, with $F_y\neq 0$, then $\frac{\partial y}{\partial x}=-\frac{F_x}{F_y}$ and $\frac{\partial y}{\partial z}=-\frac{F_z}{F_y}$.
8. **Solutions to the wave equation:** All solutions $u(t,x)$ to the Wave equation $u_{tt}=c^2u_{xx}$ are of the form $u(t,x)=g(x-ct)+h(x+ct)$, for some single-variable, twice-differentiable functions $g(y)$ and $h(y)$.
9. **Formula for directional derivatives:** $D_{\vec{u}}f(x,y)=\nabla f(x,y)\circ\vec{u}$ for any *unit* vector $\vec{u}\in\mathbb{R}^2$. This shows that the gradient $\nabla f(x,y)$ points in the direction of the fastest growth of f at (x,y) and the rate of this fastest growth is $|\nabla f(x,y)|$.
10. **Formula for tangent lines/planes of level sets:**
    - If $f(x,y)=c$ is a level curve of a differentiable function $f(x,y)$ and $P=(x_0,y_0)$ is a point on this level curve, then $\nabla f(P)\perp$ the level curve at P.

      Hence, the tangent line to the level curve at P is given by:
      $$\frac{\partial f}{\partial x}(x_0,y_0)(x-x_0)+\frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)=0.$$
    - If $f(x,y,z)$ is differentiable at $P=(x_0,y_0,z_0)$, then its tangent plane at P has normal vector $\nabla f(x_0,y_0,z_0)$; i.e., it is given by:
      $$\frac{\partial f}{\partial x}(x_0,y_0,z_0)(x-x_0)+\frac{\partial f}{\partial y}(x_0,y_0,z_0)(y-y_0)+\frac{\partial f}{\partial z}(x_0,y_0,z_0)(z-z_0)=0.$$
11. **Necessary Condition for Local Extrema:** If $f(x,y)$ is differentiable and has a local extremum at (x_0,y_0), then $\nabla f(x_0,y_0)=\langle 0,0\rangle$; i.e., $f(x,y)$ has a critical point at P.
12. **2nd Derivative Test:** If $f(x,y)$ has a critical point at (a,b), and all 4 2nd-order partial derivatives are continuous nearby (a,b), set $D=\left(f_{xx}f_{yy}-f_{xy}^2\right)\big|_{(a,b)}$.
    (a) If $D>0$ and $f_{xx}(a,b)>0$, then $f(a,b)$ is a local minimum.
    (b) If $D>0$ and $f_{xx}(a,b)<0$, then $f(a,b)$ is a local maximum.
    (c) If $D<0$, then $f(a,b)$ is a saddle point.
    (d) If $D=0$, the test fails to reach a conclusion. We need another test!
13. **Extreme Value Theorem:** If $f(x,y)$ is continuous on a **closed** and **bounded** domain D_f, then f has a global minimum and a global maximum (somewhere on D_f).
14. **Nice Domain Method** If $f(x,y)$ has continuous partial derivatives f_x and f_y on a closed and bounded domain D_f in $\mathbb{R}^2$, then the global extrema of f on D_f are among the two sources:
    (a) critical points: $\nabla f(x,y)=\vec{0}$ for some $(x,y)\in D_f$.
    (b) extrema of f along the boundary ∂D_f.
15. **Lagrange Multipliers:** If $f(x,y)$ has a global extremum along the constraint curve $g(x,y)=k$, and both functions are differentiable, with $\nabla g\neq\vec{0}$, then this global extremum is obtained at one of the solutions (x_0,y_0) of the system, where λ is called a *Lagrange multiplier*:
    $$\left|\begin{array}{rcl} f_x(x_0,y_0) &=& \lambda g_x(x_0,y_0);\\ f_y(x_0,y_0) &=& \lambda g_y(x_0,y_0);\\ g(x_0,y_0) &=& k.\end{array}\right.$$
16. **Fubini's Theorem:** If $f(x,y)$ is continuous on a rectangle $R=[a,b]\times[c,d]$ then $\iint_R f(x,y)\,dA=\int_a^b\int_c^d f(x,y)\,dy\,dx=\int_c^d\int_a^b f(x,y)\,dx\,dy$.
17. **Type I Region:** If D is of type I region in $\mathbb{R}^2$; i.e., $a\le x\le b$, and for any such fixed x, $g(x)\le y\le h(x)$, and $f(x,y)$ is a function defined on D, then
    $$\iint_D f(x,y)\ dA=\int_a^b\int_{g(x)}^{h(x)} f(x,y)\ dy\ dx.$$

3 Problem Solving Techniques

1. **Proving that a limit does not exist:** To show that a limit $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist, we could attempt to find two different paths approaching the origin such that $f(x,y)$ has different limits along those paths. Alternatively we can also try to find a path along which the limit does not exist. Some comments/tips:
   - Usually, it will be easy to compute the limits along the axes $\lim_{x\to 0} f(x,0)$ and $\lim_{y\to 0} f(0,y)$, so we should do that first and then search for a path giving a different limit.
   - Typically, the next best thing to try are the diagonals $y = x$ and $y = -x$.
   - If all these still produce the same limit, we can try a more general line $y = mx$ for some parameter m and see if for some m the limit is different from the above.

   Example: We show that $\lim_{(x,y)\to(0,0)} f(x,y)$ where $f(x,y) = \frac{x^3y - xy^3}{x^4+y^4}$ does not exist:
   First, we see that $f(x,0) = 0$, so the limit along $y = 0$ is zero. Taking the limit along $x = 0$ and $y = x$ also gives 0, so this doesn't help. Second, we try $y = mx$ and see that
   $$f(x, mx) = \frac{mx^4 - m^3x^4}{x^4 + m^4x^4} = \frac{m(1-m^2)}{1+m^4}.$$
   Evidently, $\lim_{x\to 0} f(x, mx) = \frac{m(1-m^2)}{1+m^4}$, so picking, say, $m = \frac{1}{2}$ gives a limit $\neq 0$. □

   - If $y = mx$ still doesn't work, we can try to find the limits along $y = x^\alpha$ or even $y = mx^\alpha$, where α is some parameter. It might be beneficial to guess a value of α that makes $f(x, x^\alpha)$ particularly simple.

   Example: We show that $\lim_{(x,y)\to(0,0)} f(x,y)$ where $f(x,y) = \frac{yx^2}{x^6+y^2}$ does not exist.
   We quickly see that $f(x,0) = 0$, so we need to find a path along which the limit is not 0. If we try $y = mx$, we will still get 0 as the limit, so we try $y = x^\alpha$. We observe that:
   $$f(x, x^\alpha) = \frac{x^{2+\alpha}}{x^6 + x^{2\alpha}}.$$
   Our denominator becomes quite simple if $2\alpha = 6$, so we might try $\alpha = 3$. Indeed, $f(x, x^3) = \frac{1}{2x}$ does not even have a limit as $x \to 0$. (So looking back, we wouldn't even have had to consider any other path before.) □

   - Technically, our path could be any parametric curve; e.g. $x(t) = e^{-t}\cos t, y(t) = e^{-t}\sin t$ as $t \to \infty$. However, it is mostly sufficient to consider paths of the form $y = g(x), x \to 0$ or $x = g(y), y \to 0$ for some function $g(x)$ such that $\lim_{x\to 0} g(x) = 0$.

2. The **Chain Rule:** Let $f(x,y)$ be a function of two variables x and y, which, in turn, are functions of another variable t. Then
   $$\frac{d}{dt} f(x(t), y(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$
   If x and y instead are functions of two variables s, t, we have:
   $$\frac{d}{dt} f(x(t,s), y(t,s)) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} \quad \text{and} \quad \frac{d}{ds} f(x(t,s), y(t,s)) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}.$$
   This should be thought of as a chain of events: In the latter case, we want to know how a (small) change Δs in s affects $f(x(t,s), y(t,s))$. This happens in two different ways:
   (a) A change Δs in s causes a change $\Delta x = \frac{\partial x}{\partial s}\Delta s$ in x, which, in turn, causes a change $\Delta_1 f = \frac{\partial f}{\partial x}\Delta x$ in $f(x,y)$.
   (b) At the same time, the change in s causes a change $\Delta y = \frac{\partial y}{\partial s}\Delta s$ in y and, thus, another change $\Delta_2 f = \frac{\partial f}{\partial y}\Delta y$ in $f(x,y)$.
   Therefore, the total change in $f(x,y)$ is $\Delta f = \Delta_1 f + \Delta_2 f = \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}\right)\Delta s$.

3. **Implicit Differentiation:** Let x, y and z satisfy some equation $F(x, y, z) = 0$. Suppose (see the Implicit Function Theorem earlier) that we could, in theory, solve this equation for x, obtaining x in terms of y and z. Then we could take the partial derivatives $\partial x/\partial y$ and $\partial x/\partial z$. There is a shortcut for computing this, without actually solving for x:

$$\frac{\partial x}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial x};$$
$$\frac{\partial x}{\partial z} = -\frac{\partial F/\partial z}{\partial F/\partial x}.$$

4. **Finding local mimima/maxima** of a function $f(x, y)$. This method also works when we are asked to find and classify the critical points.
   1. Compute $\nabla f(x, y)$ and find the solutions of $\nabla f(x, y) = \vec{0}$: these are the critical points.
   2. Use the **2nd derivative test** for every critical point to classify it; namely, compute $D = f_{xx}f_{yy} - f_{xy}^2$ at the critical point. If:
      - $D > 0$ and $f_{xx} > 0$, then $f(x, y)$ is a local minimum.
      - $D > 0$ and $f_{xx} < 0$, then $f(x, y)$ is a local maximum.
      - $D > 0$ and $f_{xx} = 0$, go back and redo your computations of f_{xx} and D because this never happens!
      - $D < 0$, then $f(x, y)$ is a saddle point, so it is neither a minimum nor a maximum.
      - $D = 0$, then we can't say anything about this point using the 2nd derivative test.
5. **Finding global minima/maxima on closed and bounded domains** for a function $f(x, y)$ defined on D_f. This is done in three steps:
   1. Find the critical points of f in the interior of D_f by solving $\nabla f(x, y) = \langle 0, 0\rangle$. (There might be solutions that don't lie in D_f but we ignore them.) Classifying the criticial with the second derivative test (or otherwise) is not necessary here.
   2. Find the extrema of f along the boundary ∂D_f. Sometimes this requires breaking the boundary up into pieces where we can either reduce $f|_{\partial D_f}$ to a single-variable calculus problem (e.g., parametrize lines or circles and plug them into f) or we can use the method of Lagrange Multipliers.
   3. Compute the value of f on all the thus-found critical points in the interior and extrema along the boundary, and pick the maxima and minima among them.
6. **Lagrange Multipliers:** We want to maximize/minimize a function $f(x, y)$ on a curve described by an equation $g(x, y) = k$.
   1. Compute ∇f and ∇g.
   2. Solve the system of equations $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = k$ for x, y, and λ (we only need the solutions for x and y).
   3. Compute the value of $f(x, y)$ for every solution and decide which one(s) are the maxima/minima.

4 Problems for Review

The exam will be based on Homework, Lecture, Section and Quiz problems. Review **all** homework problems, and all your classnotes and discussion notes. Such a thorough review should be enough to do well on the exam. If you want to give yourself a mock-exam, select 4 representative problems from various HW assignments, give yourself 40 minutes, and then compare your solutions to the VW solutions. If you didn't manage to do some problems, analyze for yourself what went wrong, which areas, concepts and theorems you should study in more depth, and if you ran out of time, think about how to manage your time better during the upcoming exam.

4.1 Limits and Continuity

1. True/False practice:
   (a) If f is a function whose domain contains points arbitrarily close to $(2,3)$, then $\lim_{(x,y)\to(2,3)} f(x,y) = (2,3)$.

   Solution: False. This doesn't even make sense because f is single-valued.

   (b) If $\lim_{(x,y)\to(a,b)}$ exists, the function $f(x,y)$ must be defined or continuous at (a,b).

   Solution: False. We can for example define $f(x,y)$ to be zero when $(x,y) \neq (0,0)$ and to be 1 when $(x,y) = (0,0)$. If we then pick $(a,b) = (0,0)$ the limit of $f(x,y)$ exists for $(x,y) \to (a,b)$ but it is not equal to $f(a,b)$. We could also leave $f(x,y)$ undefined at the origin to give a counterexample to the first statement.

   (c) The function $f(x,y) = x - y + 1$ is not continuous at the point $(0,1)$.

   Solution: False. This function is linear, so it is continuous.

   (d) To show that the limit at a point (a,b) exists, it suffices to find two paths to the point (a,b) where the limits of $f(x,y)$ agree.

   Solution: False. We would have to show the limit agrees on every path, but this is not feasible.

   (e) To prove that the limit of $f(x,y)$ at $(x,y) \to (a,b)$ does not exist, we have to prove that the limits along at least 3 different paths are different.

   Solution: False. It is sufficient to show that the limits along two different paths are different.

   (f) If $f(x,y) \to L$ as $(x,y) \to (a,b)$ along any line through (a,b), then $\lim_{(x,y)\to(a,b)} f(x,y) = L$.

   Solution: False. One counterexample is

   $$f(x,y) = \begin{cases} 0 & \text{if } y \leq 0 \text{ or } y \leq 2x^2 \\ 1 & \text{if } 0 < y < 2x^2 \end{cases}$$

   Along every straight line passing through the origin this will be constant 0 near the origin, but the limit approaching via the parabola $y = x^2$ is 1.

   (g) The (ϵ, δ)-definition of limits and continuity can be extended to functions of 3 and more variables.

   Solution: True. We just need to replace $\sqrt{(x-a)^2 + (y-b)^2}$ by $\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$ etc.

   (h) If f and g are two continuous functions in $\mathbb{R}$, $(a,b) \in \mathbb{R}^2$ and $\lim_{x\to a} f(x) = L_1$ and $\lim_{y\to b} g(y) = L_2$ then $\lim_{(x,y)\to(a,b)} f(x)g(y) = L_1 L_2$.

   Solution: True. $h_1(x,y) = f(x)$ and $h_2(x,y) = g(y)$ are continuous functions so their product $h_1(x,y)h_2(x,y) = f(x)g(y)$ is continuous.

2. Show that the limit does not exist.

(a) $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^4+xy^3}$;

Solution: Approaching the origin along the diagonal $x = y$ gives a limit

$$\lim_{x\to 0} \frac{x^3}{2x^4} = \lim_{x\to 0} \frac{1}{2x}$$

which does not exist

(b) $\lim_{(x,y)\to(1,0)} \frac{x+y^2}{(x-1)^3+y^3}$.

Solution: Approaching the limit along the x-axis, i.e. $y = 0$ we need to consider

$$\lim_{x\to 1} \frac{x}{(x-1)^3}$$

which does not exist since the numerator has limit 1 but the denominator has limit 0.

3. Discuss continuity of the following functions:

(a) $\frac{2x^4y}{x^8+y^2}$;

Solution: This is obviously continuous whenever the denominator is nonzero, i.e. everywhere except at the origin. But at the origin it is not continuous (or rather we can't extend the function to the origin in a way which makes it continuous): If we approach the origin via any of the coordinate axes we get a limit of 0, but if we approach it via the path $y = x^4$ we get a limit

$$\lim_{x\to 0} \frac{2x^4x^4}{x^8+x^8} = 1$$

Hence the limit $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist.

(b) $\frac{x^2\cos^2 x}{x^2+(y-1)^2}$.

Solution: This is continuous when the denominator is nonzero, so everywhere except at $(0, 1)$. At $(0, 1)$ it can't be defined in a ways which makes the function continuous: Approaching $(0, 1)$ via the y-axis, we get the limit

$$\lim_{y\to 1} \frac{0\cos^2(0)}{(y-1)^2} = 0$$

On the other hand if we approach $(0, 1)$ on the line $y = 1$ we get the limit

$$\lim_{x\to 0} \frac{x^2\cos^2 x}{x^2} = \cos^2(0) = 1$$

4. Find the domain of the following functions and prove that they are continuous there:

(a) $f(x, y) = x^2 + 2xy + e^x - \cos y - 2$;

Solution: This is continuous because it is a sum of continuous functions.

(b) $f(x, y) = \frac{xy^2}{x^2+y^2}$ for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$. (More than one solution?!)

Solution: Away from the origin this is a quotient of continuous functions with nonzero denominator and therefore continuous.
To show continuity at the origin first observe that $|xy| \leq \frac{1}{2}\left(x^2+y^2\right)$. This inequality can be seen by applying the binomial formula to the left hand sides of $(x+y)^2 \geq 0$ and $(x-y)'2 \geq 0$. Dividing both sides of the inequality by x^2+y^2 and multiplying by $|y|$ in turn gives

$$|f(x,y)| = \frac{|xy^2|}{x^2+y^2} \leq \frac{1}{2}|y| \tag{1.1}$$

Now we use the ϵ–δ-definition of continuity to show that f is continuous. Given any $\epsilon > 0$ we choose $\delta = \frac{1}{2}\epsilon$. Now when $\sqrt{x^2+y^2} < \delta$ we must have $|y| < \delta$ and therefore $|f(x,y) - 0| < \frac{1}{2}|y| < \frac{1}{2}\delta = \epsilon$ using the inequality (1.1). This shows that the limit of $f(x,y)$ are $(x,y) \to (0,0)$ is zero and since we defined $f(0,0) = 0$, $f(x,y)$ is continuous.

4.2 Partial Derivatives

1. True/False practice:
   (a) Clairaut's Theorem says that $f_{xy} = f_{yx}$.

   Solution: True, but Clairaut's theorem also has a prerequisite: All second partial derivatives of f, that is f_{xx}, f_{xy}, f_{yx} and f_{yy} need to be continuous.

   (b) $f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$.

   Solution: True. The shorthand f_x means $\frac{\partial f}{\partial x}$ and hence $f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x}$.

   (c) Any second partial derivative of a sum is the sum of the corresponding second partial derivatives, assuming all these derivatives exist:

   $$\frac{\partial^2 (f_1+f_2)}{\partial w^2} = \frac{\partial^2 f_1}{\partial w^2} + \frac{\partial^2 f_2}{\partial w^2}.$$

   Solution: True. A (partial) derivative of a sum of two functions is the sum of the (partial) derivatives, so using this fact proves this statement.

2. Implicit differentiation:
   (a) Find $\frac{\partial y}{\partial x}$ if $x^2 + 2xy^2 + z^3 + xyz + y = 2$.

   Solution: Define $F(x,y,z) = x^2 + 2xy^2 + x^3 + xyz + y - 2$ so we can write our equation as $F(x,y,z) = 0$. Then by the formula from the lecture

   $$\begin{aligned}\frac{\partial y}{\partial x} &= -\frac{\partial F/\partial x}{\partial F/\partial y}\\ &= -\frac{2x + 2y^2 + 3x^2 + yz}{4xy + xz + 1}\end{aligned}$$

   (b) Find $\frac{\partial x}{\partial y}$ for the above equation.

   Solution: $\partial x/\partial y$ is just the reciprocal of $\partial y/\partial x$ so

   $$\frac{\partial x}{\partial y} = -\frac{4xy + xz + 1}{2x + 2y^2 + 3x^2 + yz}$$

(c) Find $\frac{\partial y}{\partial x}$ for $e^y \sin x = x + xy$.

Solution: As before define $F(x,y) = e^y \sin x - x - xy$. Then

$$\begin{aligned}\frac{\partial y}{\partial x} &= -\frac{\partial F/\partial x}{\partial F/\partial y} \\ &= -\frac{e^y \cos x - 1 - y}{e^y \sin x - x}\end{aligned}$$

3. Higher order computation:

(a) Prove that $c(x,t) = \frac{1}{\sqrt{Dt}} e^{-\frac{x^2}{4} Dt}$ is a solution of the diffusion equation $\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2}$.

Solution:

$$\begin{aligned}\frac{\partial c}{\partial t} &= -\frac{1}{2\sqrt{D}\, t^{3/2}} e^{-x^2/4Dt} + \frac{1}{\sqrt{Dt}} \frac{x^2}{4Dt^2} e^{-x^2/4Dt} \\ \frac{\partial c}{\partial x} &= -\frac{x}{2Dt} \frac{1}{\sqrt{Dt}} e^{-x^2/4Dt} \\ \frac{\partial^2 c}{\partial x^2} &= -\frac{1}{2Dt\sqrt{Dt}} e^{-x^2/4Dt} + \frac{x^2}{4D^2t^2} \frac{1}{\sqrt{Dt}} e^{-x^2/4Dt}\end{aligned}$$

Solution: We see that the RHS of the third equation is D times the RHS of the first, showing that c satisfies the diffusion equation.

(b) *(Computation avid?)* Find $\frac{\partial^2 c}{\partial t^2}$ for the above function $c(x,t)$.

Solution:

$$\begin{aligned}\frac{\partial^2 c}{\partial t^2} &= \frac{3}{4\sqrt{D}t^{5/2}} e^{-x^2/4Dt} - \frac{x^2}{8D\sqrt{D}t^{7/2}} e^{-x^2/4Dt} \\ &\quad - \frac{5x^2}{8D\sqrt{D}t^{7/2}} e^{-x^2/4Dt} + \frac{x^4}{16\sqrt{D}D^2t^{9/2}} \\ &= \frac{12D^2t^2 - 12Dtx^2 + x^4}{16D^{5/2}t^{9/2}} e^{-x^2/4Dt}\end{aligned}$$

4. The *Van der Waals* equation equation of state for a gas is $\left(p + \frac{n^2 a}{V^2}\right)(V - nb) = nRT$, where p is the pressure, V the volume, T the temperature, n the amount of moles in the gas, and R, a, b are positive constants. We can always assume that $V > nb$.

(a) Calculate $\frac{\partial T}{\partial p}$ and $\frac{\partial T}{\partial V}$.

Solution: We can easily solve the equation for T obtaining $T = \frac{1}{nR}\left(p + \frac{n^2 a}{V^2}\right)(V - nb)$. Now we compute

$$\begin{aligned}
\frac{\partial T}{\partial p} &= \frac{V - nb}{nR} = \frac{V}{nR} - \frac{b}{R} \\
\frac{\partial T}{\partial V} &= \frac{1}{nR}\left(p + \frac{n^2 a}{V^2}\right) - 2\frac{V - nb}{nR}\frac{n^2 a}{V^3} \\
&= \frac{p}{nR} + \frac{a(2nb - V)n}{RV^3}
\end{aligned}$$

(b) Give the linear approximation of T for a small increase of p and V.

Solution:

$$\begin{aligned}
T(p + \Delta p, V + \Delta V) &= T(p, V) + \frac{\partial T}{\partial p}(p, V)\Delta p + \frac{\partial T}{\partial V}(p, V)\Delta V \\
&= T(p, V) + \frac{V - nb}{nR}\Delta p + \frac{1}{nR}\left(p + \frac{an(2nb - V)}{RV^3}\right)\Delta V
\end{aligned}$$

(c) Find the *critical point* of a Van der Waals gas (p_c, V_c) at which $\frac{\partial p}{\partial V} = \frac{\partial^2 p}{\partial V^2} = 0$. (This is a challenging computation!)

Solution: We can use implicit differentiation to find $\partial p/\partial V$: Define $F(p,V,T) = \left(p + \frac{n^2a}{V^2}\right)(V - nb) - nRT$. Then

$$\begin{aligned}\frac{\partial p}{\partial V} &= -\frac{\partial F/\partial V}{\partial F/\partial p}\\ &= \frac{p + an^2(2nb - V)/V^3}{nb - V}\\ &= \frac{pV^3 + an^2(2nb - V)}{V^3(nb - V)} \qquad (2.2)\end{aligned}$$

Note that the computation of the partial derivatives of F is almost the same as that of the partial derivatives of T.
For now let us denote the numerator of (2.2) by $N(p,V)$ and the denominator by $D(p,V)$. We need to find solutions of $N(p_c,V_c)/D(p_c,V_c) = 0$, or equivalently

$$N(p_c, V_c) = 0 \qquad (2.3)$$

that also satisfy

$$0 = \frac{\partial^2 p}{\partial V^2} = \frac{\frac{\partial N}{\partial V}D - N\frac{\partial D}{\partial V}}{D^2} \qquad (2.4)$$

Equation (2.4) is equivalent to $\frac{\partial N}{\partial V}D - N\frac{\partial D}{\partial V} = 0$ which simplifies to

$$\frac{\partial N}{\partial V} = 0 \qquad (2.5)$$

using equation (2.3).
Writing this out we obtain

$$\begin{aligned} p_c{V_c}^3 &= an^2(V_c - 2nb) && \text{(from (2.3))}\\ 3p_c{V_c}^2 &= an^2 && \text{(from (2.5))}\end{aligned}$$

and find the solution $V_c = 3nb, p_c = 27a/b^2$.

4.3 Tangent Planes and Linear Approximations

1. True/False practice:
   (a) The linear approximation $L_{(a,b)}(x,y)$ of a function $f(x,y)$ is always a good way to approximate the function around (a,b).

   Solution: True if the function is differentiable, but False in general.

2. Prove that if f is a function of two variables that is differentiable at (a,b), then f is continuous at (a,b). (Hint: go back to the definitions!)

Solution: Recall that f is differentiable at (a,b) if

$$f(a+\Delta x, b+\Delta y) = f(a,b) + f_x(a,b)\Delta x + f_y(a,b)\Delta y + \epsilon_1(\Delta x,\Delta y)\Delta x + \epsilon_2(\Delta x,\Delta y)\Delta y$$

for some continuous functions ϵ_1 and ϵ_2 such that $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as $\Delta \to 0$ and $\delta \to 0$. Using this we see that $f(x,y)$ is continuous at (a,b):

$$\begin{aligned}\lim_{(x,y)\to(a,b)} f(x,y) &= \lim_{(\Delta x,\Delta y)\to(0,0)} f(a+\Delta x, b+\Delta y)\\ &= \lim_{(\Delta x,\Delta y)\to(0,0)} \big(f(a,b) + f_x(a,b)\Delta x + f_y(a,b)\Delta y + \epsilon_1(\Delta x,\Delta y)\Delta x + \epsilon_2(\Delta x,\Delta y)\Delta y\big)\\ &= f(a,b) + \lim_{(\Delta x,\Delta y)\to(0,0)} \big(\epsilon_1(\Delta x,\Delta y)\Delta x + \epsilon_2(\Delta x,\Delta y)\Delta y\big)\\ &= f(a,b)\end{aligned}$$

3. Find the equation of the tangent plane to $z = f(x,y) = x^2\cos(\pi y) - \frac{6}{xy^2}$ at $(2,-1)$.

Solution: The normal vector to the tangent plane is given by

$$\begin{aligned}\nabla f(2,-1) &= \left\langle 2\cdot 2\cos(-\pi) + \frac{6}{2^2(-1)^2}, -\pi 2^2\sin(-\pi) + 2\frac{6}{2(-1)^3}\right\rangle\\ &= \langle -5/2, -6\rangle\end{aligned}$$

We also see that $f(2,-1) = -7)$ so the equation for the plane is $z + 7 = -\frac{5}{2}(x-2) - 6(y+1)$.

4. Find the linear approximation to $z = \cos(\sin y - x)$ at $(-2,0)$.

Solution: First we compute the partial derivatives of z:

$$\begin{aligned}\frac{\partial z}{\partial x} &= \sin(\sin y - x) = \sin(2)\\ \frac{\partial z}{\partial y} &= -\sin(\sin y - x)\cos y = -\sin(2)\end{aligned}$$

Using this we get the linear approximation $z \approx \cos(2) + \sin(2)(x+2) - \sin(2)y$

4.4 Chain Rule

1. True/False practice:
   (a) For $u = f(x,y)$, where $x = x(r,s,t), y = y(r,s,t)$, we can find $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$. (Assume all functions are differentiable.)

   Solution: True, this is the chain rule as presented in the lecture.

2. A function f is called homogeneous of degree n if it satisfies the equation

$$f(tx,ty) = t^n f(x,y) \tag{4.6}$$

for all t, where n is a positive integer and f has continuous second order derivatives.
   (a) Verify that $f(x,y) = x^2y + 2xy^2 + 5y^3$ is homogeneous of degree 3.

   Solution: $f(tx,ty) = t^2x^2ty + 2txt^2y^2 + 5t^3y^3 = t^3(x^2y + 2xy^2 + 5y^3)$

(b) Show that if f is homogeneous of degree n then $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y)$. (Hint: use the Chain Rule to differentiate $f(tx, ty)$ with respect to t.)

Solution: Using the chain rule we obtain

$$\frac{d}{dt}f(tx, ty) = \frac{\partial f}{\partial x}(tx, ty)\frac{d}{dt}tx + \frac{\partial f}{\partial y}(tx, ty)\frac{d}{dt}ty = x\frac{\partial f}{\partial x}(tx, ty) + y\frac{\partial f}{\partial y}(tx, ty) \tag{4.7}$$

On the other hand, the homogeneity condition (4.6) shows that

$$\frac{d}{dt}f(tx, ty) = \frac{d}{dt}t^n f(x, y) = nt^{n-1}f(x, y) \tag{4.8}$$

Now if we can equate the right hand sides of equations (4.7) and (4.8) and set $t = 1$ to obtain the desired equation.

(c) If f is homogeneous of degree n, show that $f_x(tx, ty) = t^{n-1}f_x(x, y)$ for $t > 0$.

Solution: We take the partial u derivative of both sides of equation $f(tu, tv) = t^n f(u, v)$ (this is just 4.6 plugging in u and v instead of x and y):

$$\begin{aligned} \frac{\partial}{\partial u}t^n f(u, v) &= t^n f_x(u, v) \\ \frac{\partial}{\partial u}f(tu, tv) &= tf_x(tu, tv) \end{aligned}$$

Hence $t^{n-1}f_x(u, v) = f_x(tu, tv)$ after canceling one t and plugging in $u = x, v = y$ shows the desired equation.

3. Let $g(s, t) = f(s^2 - t^2, t^2 - s^2)$ and f be differentiable. Prove that $t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0$.

Solution: We compute, using $x = s^2 - t^2, y = t^2 - s^2$):

$$\begin{aligned} \frac{\partial g}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = \frac{\partial f}{\partial x}2s - \frac{\partial f}{\partial y}2s \\ \frac{\partial g}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = -\frac{\partial f}{\partial x}2t + \frac{\partial f}{\partial y}2t \end{aligned}$$

4. Let $f(u - v^2, u^3 + v)$ be differentiable, and so be its derivatives. Find $\frac{\partial^2 f}{\partial u \partial v}$.

Solution: Using $x = u - v^2, y = u^3 + v$ we compute:

$$\begin{aligned} \frac{\partial f}{\partial v} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} = -f_x 2v + f_y \\ \frac{\partial^2 f}{\partial u \partial v} &= -f_{xx}2v\frac{\partial x}{\partial u} - f_{xy}2v\frac{\partial y}{\partial u} + f_{yx}\frac{\partial x}{\partial u} + f_{yy}\frac{\partial y}{\partial u} \\ &= -f_{xx}2v + f_{xy}\left(1 - 6u^2 v\right) + f_{yy}3u^2 \end{aligned}$$

4.5 Gradient Vector

1. True/False practice:

(a) The gradient vector $\nabla f(a, b)$ for a two-variable function $z = f(x, y)$ lives in 3d space and is perpendicular to the tangent plane of the graph at $(a, b, f(a, b))$.

Solution: False. The gradient vector of a n-variable function lives in n-dimensional space (so 2d space in our case). The gradient is perpendicular to the **level curve** $f(x,y) = f(a,b)$ passing through (a,b).

(b) $D_{\vec{\imath}+\vec{\jmath}} f(x,y) = f_x\vec{\imath} + f_y\vec{\jmath}$.

Solution: True. $D_{\vec{\imath}+\vec{\jmath}} f(x,y) = \nabla f(x,y) \cdot \langle 1,1\rangle = f_x\vec{\imath} + f_y\vec{\jmath}$

(c) The gradient of a function is always orthogonal to the direction of maximum change of the function.

Solution: False. The gradient vector points in the direction of maximum change and is perpendicular to all the directions of zero change.

2. Find the maximum rate of change and its direction for $f(x,y) = \sqrt{x^2+y^2}$ at $(-1,1)$.

Solution:

$$\nabla f(x,y) = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$$

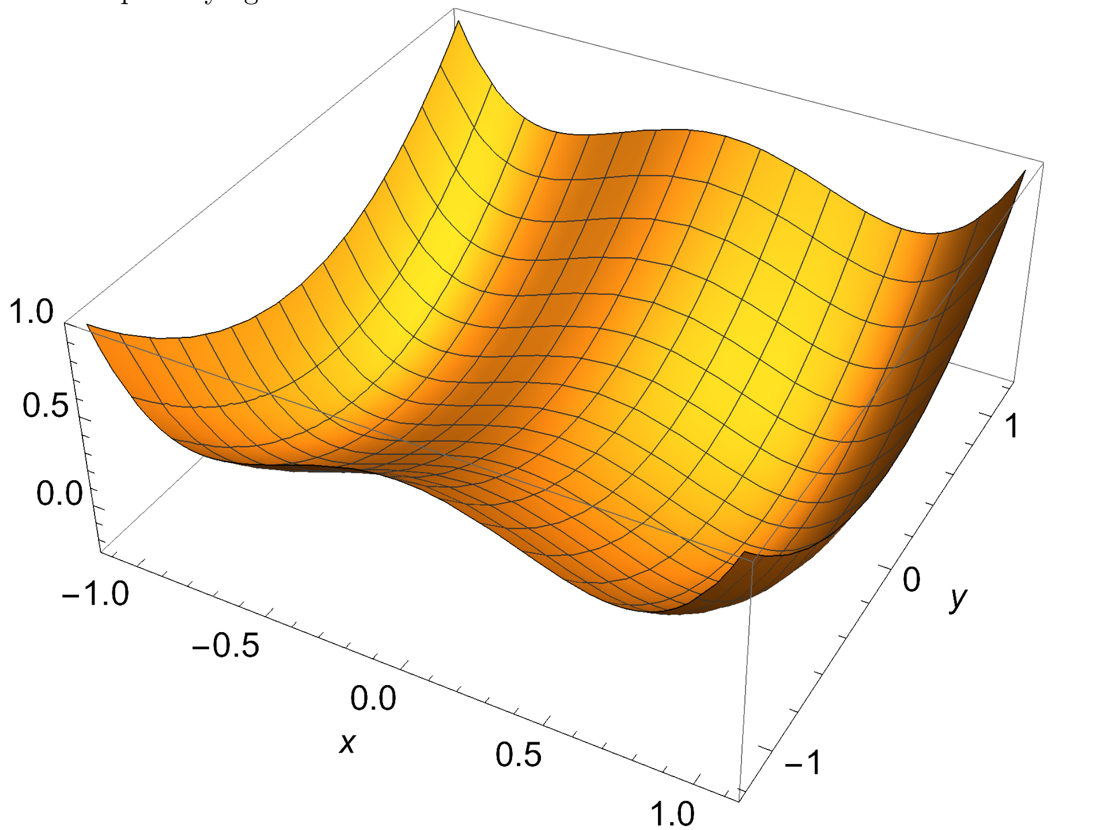
so in particular $\nabla f(-1,1) = \langle -1/\sqrt{2}, 1/\sqrt{2}\rangle$ points in the direction of maximal change. The maximum rate of change is

$$|\nabla f(-1,1)| = \sqrt{\left(\frac{-1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = 1$$

4.6 Extrema

1. True/False practice:
   (a) If $f(x,y)$ has two local maxima, then it must have a local minimum too.

Solution: False. This is true for single-variable functions but no longer holds in 2d. For example, the function $f(x,y) = x^4 - x^2 + \frac{1}{2}y^2$ depicted below has two minima with a saddle point lying between them.



(b) The normal vector to the surface $z = f(x,y)$ at the point $(a, b, f(a,b))$ is

$$\langle f_x(a,b), f_y(a,b), -1\rangle.$$

Solution: True. This surface is described by the equation $f(x,y) - z = 0$ whose is $\langle f_x(a,b), f_y(a,b), -1\rangle$.

(c) To find the maximum of $f(x,y)$, one simply needs to find the points (a,b) at which $f_x(a,b) = 0$ and $f_y(a,b) = 0$.

Solution: False. These points could also be maxima or saddle points, for example.

(d) Suppose the 2nd partial derivatives of D is continuous on a disk near (a,b). Then using the 2nd derivative test, if the determinant $D > 0$ and $f_{yy}(a,b) > 0$, we cannot determine if this is a local minimum or maximum because we do not know the sign of $f_{xx}(a,b)$.

Solution: False. The second derivative test **does** tell us that (a,b) is a minimum in this case. And actually we **do** know that $f_{xx}(a,b)$ is positive since $0 < D = f_{xx}f_{yy} - f_{xy}^2$ implies $f_{xx} > f_{xy}^2/f_{yy} > 0$.

(e) The normal vector to the surface $z = f(x,y)$ is three-dimensional, while the normal vector to the level curve of $z = f(x,y)$ is two-dimensional.

Solution: True. The surface $z = f(x,y)$, i.e. the graph of $f(x,y)$ lives in three-dimensional space while the level curve lives in two-dimensional space.

(f) The graph of $f(x, y) = x^2 - xy - y^2$ has a saddle point.

Solution: We have $\nabla f(x, y) = \langle 2x - y, -2y - x\rangle$ which is only zero at $(0, 0)$. At that point we obtain $f_{xx} = 2, f_{yy} = -2, f_{xy} = -1$, so $D = f_{xx}f_{yy} - {f_{xy}}^2 = -5 < 0$ and by the second derivative test $(0, 0)$ is a saddle point.

2. Find a classify all the critical points of the following functions:

(a) $f(x, y) = 7x - 8y + 2xy - x^2 + y^2$;

Solution: We need to solve $\nabla f(x, y) = \langle 7 + 2y - 2x, -8 + 2x + 2y\rangle = \langle 0, 0\rangle$. Adding the two equations give $4y - 1 = 0$, so $y = \frac{1}{4}$ and from this we see that $x = \frac{15}{4}$. The second derivatives of f are $f_{xx} = -2, f_{yy} = 2, f_{xy} = 2$, so $D = -8 < 0$ at every point. In particular, our critical point is a saddle point.

(b) $f(x, y) = (3x + 3x^3)(y^2 + 2y)$;

Solution: We first find the solutions of $\nabla f(x, y) = \langle (y^2+2y)(2+9x^2), (3x+3x^3)(2y+2)\rangle = \langle 0, 0\rangle$. Since $9x^2+2$ is always positive, we must have $y^2+2y = 0$, so either $y = 0$ or $y = -2$ in order to satisfy the first equation. Both of these values for y give $2y+2 \neq 0$, so the second equation force $3x + 3x^3 = 0$, i.e. $x = 0$. Hence our critical points are $(0, 0)$ and $(0, -2)$.
The second derivatives of f are

$$f_{xx} = 18x$$
$$f_{xy} = 2y + 2$$
$$f_{yy} = 2$$

We see that $D(0, 0) = -4 = D(0, -2)$, so both of these are saddle point.

(c) $f(x, y) = (y - 2)x^2 - y^2$;

Solution: Solutions of $\nabla f(x, y) = \langle 2x(y - 2), x^2 - 2y\rangle = \langle 0, 0\rangle$ must satisfy $x^2 = 2y$ and either $x = 0$ or $y = 2$. In the first case we get $(0, 0)$, the second one becomes $(2, 2)$. The second derivatives of f are

$$f_{xx} = 2y - 4$$
$$f_{xy} = 2x$$
$$f_{yy} = -2$$

hence we obtain $D(0, 0) = 8 > 0, f_{xx}(0, 0) = -4 < 0$ and $D(2, 2) = -16$. Using the second derivative test we see that $(0, 0)$ is a local maximum and $(2, 2)$ is a saddle point.

(d) $f(x, y) = xye^{x^2+y^2}$.

Solution: $\nabla f(x,y) = \left\langle \left(1+2x^2\right) y e^{x^2+y^2}, x\left(1+2y^2\right) e^{x^2+y^2} \right\rangle$. Since the exponential factors, as well as $1+2x^2$ and $1+2y^2$ are always positive we might as well ignore when finding the zeros. So we are left with $\langle x,y\rangle = \langle 0,0\rangle$. The second derivatives of f are

$$f_{xx} = \left(6x+4x^3\right) y e^{x^2+y^2}$$
$$f_{xy} = \left(1+2x^2\right)\left(1+2y^2\right) e^{x^2+y^2}$$
$$f_{yy} = x\left(6y+4y^3\right) e^{x^2+y^2}$$

Hence $D(0,0) = -1$, so $(0,0)$ is a saddle point.

4.7 Lagrange Multipliers

1. True/False practice:
   (a) The method of Lagrange multipliers gives us an efficient method to find the intersection between the plane $z = 2x - y + 3$ and the ellipsoid $x^2 + y^2 + z^2 = 1$.

   Solution: False. Lagrange multipliers are used to find extrema of a function satisfying a constraint but here we would need to solve a system of equations.

   (b) To find the extrema of a function via MLM, we must find (among other things) the value of the corresponding Lagrange multiplier.

   Solution: False. We don't care about the value of λ when using Lagrange multipliers.

2. Consider $f(x,y) = xy$ and $x^2 - y = 12$. We assume $y \leq 0$.
   (a) Why do we need $y \leq 0$ here?

   Solution: Else a minimum and maximum would not exist, since we could just pick y very large and $x = \pm\sqrt{y+12}$ would also have a very large magnitude, resulting in a very large or very negative value of $f(x,y)$.

   (b) Find the extreme values of f subject to the above constraints.

Solution: The shape describes the portion of the parabola $y = x^2 - 12$ in the third and fourth quadrant. This shape is closed and bounded so we are guaranteed the existence of global maxima and minima (though possibly not unique). If they lie in the part of the parabola where $y < 0$ we will find them with Lagrange multipliers. The two boundary points of this shape which occur at $y = 0$ need to be handled separately. This part is not very hard, since we immediately see that immediately see that f has value $f(0, \text{anything}) = 0$ on them.
Now we proceed to find the critical points of f on the parabola $g(x, y) = x^2 - y - 12 = 0$. The system of equations we need to solve is obtained from $\nabla f(x, y) = \langle y, x\rangle$ and $\nabla g(x, y) = \langle 2x, -1\rangle$. It is

$$\begin{aligned} y &= 2\lambda x \\ x &= -\lambda \\ x^2 - y &= 12 \end{aligned}$$

Combining the first two equations we see that $y = -2x^2$ and using the third $3x^2 = 12$, so $x = \pm 2$ and $y = -8$. Computing the value of f at these points we see that the global maximum of f is at $(2, -8)$ and has a value of 16 while the global minimum with value -16 is at $(-2, -8)$.

3. Consider $f(x, y, z) = xyz$ and $g(x, y, z) = x + y^2 + 9z^2 = 4$. We assume $x \geq 0$.
   (a) Why do we need $x \geq 0$ here?

   Solution: $x \geq 0$ forces $y^2 + 9z^2 \leq 4$ which gives us a bounded domain. Else global maxima and minima would not exist.

   (b) Find the extreme values of f subject to the above constraints.

Solution: As before we know that global maxima and minima exist because the (constrained) domain for this problem is bounded and closed. The boundary of this surface is given by those points on the surface where $x = 0$, so f is zero on the boundary. For the interior $x > 0$ of the surface we use Lagrange multipliers. First we compute $\nabla f(x, y, z) = \langle yz, xz, xy\rangle$ and $\nabla g(x, z, y) = \langle 1, 2y, 18z\rangle$. Now we nee to solve the system of equations

$$\begin{aligned} yz &= \lambda \\ xz &= 2\lambda y \\ xy &= 18\lambda z \\ x + y^2 + 9z^2 &= 4 \end{aligned}$$

Substituting the first equation into the second two we obtain $xz = 2y^2z$ and $xy = 18yz^2$. For now, let's assume that $y \neq 0$ and $z \neq 0$ and come back to those cases later. This allows us to cancel the zs and ys on both sides of these equations giving $2y^2 = x = 18z^2$. Plugging this into the constraint gives us $4y^2 = 4$ and hence $y = \pm 1, x = 2$ and $z = \pm 1/3$. The values of f at these four critical points are $f(2, 1, 1/3) = 2/3 = f(-2, 1, -1/3), f(-2, 1, 1/3) = -2/3 = f(2, 1, -1/3)$. Before we conclude that these are the maxima and minima of f with the given constraints we need to go back to the cases where $y = 0$ or $z = 0$. But if any variable is zero then $f(x, y, z) = 0$, so these points can't be maxima or minima (since we've already found points with larger and smaller values).

4.8 Integrals

1. True/False practice:
   (a) For a continuous function f, the value $\iint_R f(x, y)\, dA$ can be viewed as a volume.

   Solution: Yes, this is the volume below the graph of f restricted to R.

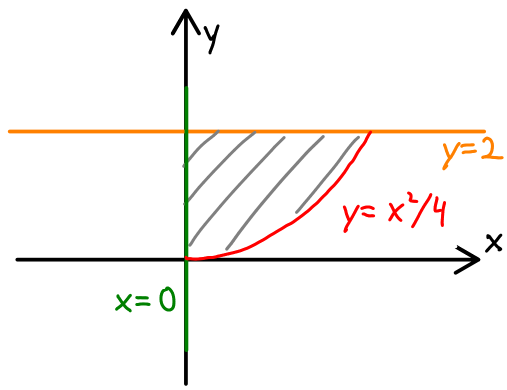
   (b) For a continuous function f, $\int_0^1 \int_0^y f(x, y)\, dx\, dy = \int_0^1 \int_0^x f(x, y)\, dy\, dx$.

   Solution: False. The integral on the left is taken over the region $0 \leq x \leq 1, 0 \leq y \leq x$ while the second one is taken over the region $0 \leq x \leq y, 0 \leq y \leq 1$. These are different so these integral won't be the same in general. An explicit example is $f(x, y) = x$: The LHS evaluates to $\frac{1}{6}$ while the R'S evaluates to $\frac{1}{3}$.

   (c) $\iint_D f(x, y)\, dA = \iint_{D_1} f(x, y)\, dA + \iint_{D_2} f(x, y)\, dA$ if $D = D_1 \cup D_2$.

   Solution: False. This only works if D_1 and D_2 overlap at most along their boundary.

2. Let $I = \iint_D 5x^3 \cos(y^3)\, dA$ where D is the region bounded by $y = 2, y = x^2/4$ and $x \geq 0$.
   (a) Make a quick sketch of the area of interest.



Solution:

(b) Evaluate I on D.

Solution: $\iint_D 5x^3\cos(y^3)\,dA = \int_0^2\int_0^{2\sqrt{y}} 5x^3\cos(y^3)\,dx\,dy = \int_0^2 20\cos(y^3)y^2\,dy = \int_0^8 \frac{20}{3}\cos u\,du = \frac{20}{3}\sin(8)$

3. Find the volume enclosed under the plane $3x+2y-z=0$ and above the region between by the parabolas $y=x^2$ and $x=y^2$.

Solution: This plane is the graph of the function $f(x,y)=3x+2y$ hence the volume is

$$\int_0^1\int_{x^2}^{\sqrt{x}} 3x+2y\,dy\,dx = \int_0^1 3x(\sqrt{x}-x^2)+x^4-x\,dx = 3(2/5-1/4)+1/5-1/2 = 3/4$$

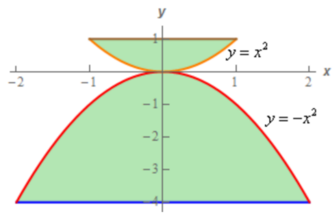
4. Evaluate the following integrals over the following regions:

(a) $\iint_D x(y-1)\,dA$, where D is bounded by $y=1-x^2$ and $y=x^2-3$.

Solution: We first need to find the points where the curves $y=1-x^2, y=x^2-3$ intersect. These are the solutions of $1-x^2=x^2-3$ that is $x=\pm\sqrt{2}$. Now we can integrate

$$\begin{aligned}\int_{-\sqrt{2}}^{\sqrt{2}}\int_{x^2-3}^{1-x^2}(1+x)y\,dy\,dx &= \int_{-\sqrt{2}}^{\sqrt{2}}(1+x)\frac{1}{2}((1-x^2)^2-(x^2-3)^2)\,dx\\ &= \int_{-\sqrt{2}}^{\sqrt{2}}(2x^3+2x^2-4x-4)\,dx\\ &= \frac{1}{2}x^4+\frac{2}{3}x^3-2x^2-4x\Big|_{-\sqrt{2}}^{\sqrt{2}}\\ &= \frac{8}{3}\sqrt{2}-8\sqrt{2} = -\frac{16\sqrt{2}}{3}\end{aligned}$$

(b) $\iint_D 3-6xy\,dA$, where D is shown below.



Solution: It is easiest to break D up into the part D_1 above the x-axis and the part D_2 below. Then

$$\begin{aligned}\iint_{D_1} 3-6xy\,dA &= \int_{-1}^{1}\int_{x^2}^{1} 3-6xy\,dy\,dx\\ &= \int_{-1}^{1} 3(1-x^2)-3x+3x^5\,dx\\ &= 6-6/3-0+0=4\end{aligned}$$

and

$$\begin{aligned}\iint_{D_2} 3-6xy\,dA &= \int_{-2}^{2}\int_{-4}^{-x^2} 3-6xy\,dy\,dx\\ &= \int_{-2}^{2} 3(4-x^2)-3x^5+48x\,dx\\ &= 48-16-0+0=32\end{aligned}$$

Hence $I=32+4=36$.

4.9 Old Math 53 Exam Problems

1. Do the following limits exist? If a limit exists, explain why and find the limit. If a limit does not exist, explain why not. (Hint: The (ϵ,δ)-definition doesn't have to be mentioned here.)
   (a) $\lim_{(x,y)\to(0,0)} \frac{x^2-y}{\sqrt{x^2+y^2}}$.

Solution: The function $g(x,y) = \frac{x^2-y}{\sqrt{x^2+y^2}}$ is continuous on its domain, i.e., on all of $\mathbb{R}^2$ except for the origin $(0,0)$ (why?). Unfortunately, our limit asks for $(x,y) \to (0,0)$, so we cannot use "continuity" reasoning to find the limit. Equally, we **cannot** say that the limit does not exist just because $(0,0)$ is not in the domain of $g(x,y)$. We need a different way of deciding if the limit exists or not. If the limit exists, then along all curves approaching $(0,0)$ that limit will have to be the same. Let's try the positive x-axis, i.e., $(x,y) = (x,0)$ with $x > 0$:

$$\lim_{(x,0)\to(0,0)} \frac{x^2-0}{\sqrt{x^2+0^2}} = \lim_{x\to 0} \frac{x^2}{\sqrt{x^2}} = \lim_{x\to 0} \frac{x^2}{|x|} \overset{|x|=x>0}{=} \lim_{x\to 0} \frac{x^2}{x} = \lim_{x\to 0} x = 0.$$

Now let's try the positive y-axis, i.e., $(x,y) = (0,y)$ with $y > 0$:

$$\lim_{(0,y)\to(0,0)} \frac{0^2-y}{\sqrt{0^2+y^2}} = \lim_{y\to 0} \frac{-y}{\sqrt{y^2}} = \lim_{y\to 0} \frac{-y}{|y|} \overset{|y|=y>0}{=} \lim_{y\to 0} \frac{-y}{y} = \lim_{y\to 0}(-1) = -1.$$

Since $0 \neq -1$, the given limit does not exist.

(b) $\lim_{(x,y)\to(\pi,\pi)} \frac{\cos(x+y)}{x+y}$.

Solution: The function $f(x,y) = \frac{\cos(x+y)}{x+y}$ is continuous on its domain, i.e., on all of $\mathbb{R}^2$ except for the line $y = -x$. This is true because the function is a quotient of the trigonometric function $\cos(x+y)$, which is continuous everywhere, and the polynomial $x+y$, which is continuous everywhere but attains 0 value exactly where $y = -x$. The point (π,π) is in the domain of the function, hence the function is continuous at (π,π) and its value there equals the desired limit:

$$\lim_{(x,y)\to(\pi,\pi)} \frac{\cos(x+y)}{x+y} \overset{\text{cont.}}{=} f(\pi,\pi) = \frac{\cos(\pi+\pi)}{\pi+\pi} = \frac{\cos(2\pi)}{2\pi} = \frac{1}{2\pi}.$$

2. A mountain lion runs on a mountain whose height above the point (x,y) is $z = x^2 + \sin^2(xy)$.

(a) In which direction(s) should the mountain lion run from point $(1,0,1)$ so that the height is increasing at the fastest possible rate? What is this fastest rate?

Solution: The fastest possible rate of increase is attained in the direction of the gradient:

$$\nabla z = \langle \partial z/\partial x, \partial z/\partial y \rangle = \langle 2x + 2\sin(xy)\cos(xy)y, 2\sin(xy)\cos(xy)x \rangle.$$

At $(1,0,1)$, this gradient is simply $\nabla z(1,0) = \langle 2,0 \rangle$, and the fastest possible rate of change is the length of the gradient, i.e., $|\nabla z(1,0)| = |\langle 2,0 \rangle| = 2$.

(b) In which direction(s) should the mountain lion run from point $(1,0,1)$ so that the height is increasing at **half** of the fastest possible rate?

Solution: Half of the fastest possible rate of height increase at point $(1,0,1)$ is $\frac{1}{2}\cdot 2 = 1$. Let $\vec{u} = \langle a, b\rangle$ be a unit vector in the desired direction. Then the directional derivative $D_{\vec{u}}(1,0) = 1$ is the desired rate of change of height at $(1,0,1)$ in the direction of $\vec{u}$; it can be calculated using the dot product with the gradient:

$$1 = D_{\vec{u}}(1,0) = \vec{u} \circ \nabla z(1,0) = \langle a, b\rangle \circ \langle 2, 0\rangle = 2a + 0b = 2a.$$

Thus, $a = 1/2$ and b must be such that $\vec{u} = \langle 1/2, b\rangle$ is unit, i.e., $(1/2)^2 + b^2 = 1$, from which $b^2 = 3/4$ and $b = \pm\sqrt{3}/2$. Therefore, the desired directions in which the height is increasing at a rate equal to half of the fastest possible rate are $\vec{u}_1 = \langle 1/2, \sqrt{3}/2\rangle$ and $\vec{u}_2 = \langle 1/2, -\sqrt{3}/2\rangle$.

3. Find the absolute maximum and minimum values attained by the function

$$f(x, y) = xy + 12(x + y) - (x + y)^2$$

on the triangle between lines $x = 0$, $y = 0$, and $x + y = 10$.

Solution: The region D in the xy-plane where the function is defined is the triangle with vertices $(0,0)$, $(10,0)$ and $(0,10)$ (see the figure). The function $f(x,y)$ is continuous on D (and on all of $\mathbb{R}^2$), as it is a polynomial in two variables. The region D is closed and bounded. Hence, by EVT, $f(x,y)$ will have global extrema on D. To find them, we first find the critical points in the interior of D:

$$\left| \begin{array}{l} \frac{\partial f}{\partial x} = y + 12 - 2(x+y) = 12 - 2x - y = 0 \\ \frac{\partial f}{\partial x} = x + 12 - 2(x+y) = 12 - x - 2y = 0 \end{array} \right. \Rightarrow \left| \begin{array}{l} y - x = 0 \\ 3(x+y) = 24 \end{array} \right. \Rightarrow \left| \begin{array}{l} y = x \\ x + y = 8 \end{array} \right.$$

From here, $x = y = 4$ and the only critical point inside D is $(4,4)$ with $f(4,4) = 16 + 12 \cdot 8 - 8^2 = 48$. Note that $(4,4)$ is indeed inside D, as $4 \geq 0$, and $4 + 4 = 8 \leq 10$. To find the critical points on the boundary ∂D of D, note that the three sides of the triangle are the components of ∂D. Restricting to each side reduces the problem to one-variable calculus:

(a) Along $y = 0$, we have $f(x,0) = 12x - x^2 = x(12-x)$ for $x \in [0,10]$. As $x(12-x)$ is a continuous function on a closed and finite interval, it attains a minimum and a maximum along $[0,10]$. These occur among the critical points inside $(0,10)$: where $\frac{d}{dx}(12x - x^2) = 12 - 2x = 0$, i.e., at $x = 6 \in (0,10)$; OR among the endpoints at $x = 0$ and $x = 10$. The corresponding values of the function are $f(6,0) = 36$, $f(0,0) = 0$, $f(10,0) = 20$.

(b) As the function and the region are symmetric with respect to variables x and y, along side $x = 0$, the function $f(0,y) = y(12-y)$ for $y \in [0,10]$ will have its global extrema among the following possibilities: $f(0,6) = 36$, $f(0,0) = 0$, $f(0,10) = 20$.

(c) Along the third side of the triangle, $x + y = 10$, we can eliminate one variable $y = 10 - x$ to obtain $f(x, 10-x) = x(10-x) + 12 \cdot 10 - 10^2 = -x^2 + 10x + 20$ for $x \in [0,10]$. Again, this is a continuous function on a closed and finite interval, and hence it attains a minimum and a maximum along $[0,10]$. The critical points inside $(0,10)$ occur where $\frac{d}{dx}(-x^2 + 10x + 20) = -2x + 10 = 0$, i.e., at $x = 5 \in (0,10)$ (and $y = 10 - 5 = 5$), with value $f(5,5) = 45$. The endpoints of $[0,10]$ yield the previously found $f(0,10) = 20 = f(10,0)$.

To summarize, $f(x,y)$ attains its maximum on D at $(4,4)$, which is inside the triangle D; the maximum value of f is $f(4,4) = 48$. Further, $f(x,y)$ attains its minimum on D at $(0,0)$, which is along the boundary of the triangle D; the minimum value of f is $f(0,0) = 0$.

4. It is known that an ellipsoid given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ (with $a, b, c > 0$) has volume $\frac{4}{3}\pi abc$. Consider all such ellipsoids that pass through point $(3,2,1)$, and among them, find the ellipsoid enclosing the **least** volume.

Solution: This problem evidently calls for Lagrange multipliers. The function we want to minimize is $f(a,b,c) = \frac{4}{3}\pi abc$. This function is continuous everywhere on $\mathbb{R}^3$; it is also bounded from below because the volume $f(a,b,c) \geq 0$. Such functions always have a global minimum. The constraint equation is

$$g(a,b,c) = \frac{9}{a^2} + \frac{4}{b^2} + \frac{1}{c^2} = 1.$$

As none of a, b, or c can be 0 (why?), the partial derivatives of g are always defined, and actually never equal 0 (as the system below shows). Thus, we can apply the method of Lagrange multipliers to $f(a,b,c)$ along the constraint $g(a,b,c) = 1$:

$$\left| \begin{array}{l} \dfrac{\partial f}{\partial a} = \lambda \dfrac{\partial g}{\partial a} \\ \dfrac{\partial f}{\partial b} = \lambda \dfrac{\partial g}{\partial b} \\ \dfrac{\partial f}{\partial c} = \lambda \dfrac{\partial g}{\partial c} \\ g(a,b,c) = 1 \end{array} \right. \Rightarrow \left| \begin{array}{l} \dfrac{4\pi}{3} bc = -\lambda \dfrac{18}{a^3} \\ \dfrac{4\pi}{3} ac = -\lambda \dfrac{8}{b^3} \\ \dfrac{4\pi}{3} ab = -\lambda \dfrac{2}{c^3} \\ \dfrac{9}{a^2} + \dfrac{4}{b^2} + \dfrac{1}{c^2} = 1 \end{array} \right. \Rightarrow \left| \begin{array}{l} \dfrac{4\pi}{3} a^3 bc = -18\lambda \\ \dfrac{4\pi}{3} ab^3 c = -8\lambda \\ \dfrac{4\pi}{3} abc^3 = -2\lambda \\ \dfrac{9}{a^2} + \dfrac{4}{b^2} + \dfrac{1}{c^2} = 1 \end{array} \right. \Rightarrow \left| \begin{array}{l} \dfrac{a^2}{b^2} = \dfrac{9}{4} \\ \dfrac{b^2}{c^2} = 4 \\ \dfrac{9}{a^2} + \dfrac{4}{b^2} + \dfrac{1}{c^2} = 1 \end{array} \right.$$

The last simplification was obtained by dividing the first by the second equation, and then the second by the third equation. We now obtain that $\frac{9}{a^2} = \frac{4}{b^2}$ and $\frac{1}{c^2} = \frac{4}{b^2}$. Substituting into the constraint:

$$\frac{4}{b^2} + \frac{4}{b^2} + \frac{4}{b^2} = 1 \;\Leftrightarrow\; \frac{12}{b^2} = 1 \;\Leftrightarrow\; b^2 = 12 \;\Leftrightarrow\; b = \sqrt{12} = 2\sqrt{3}.$$

From here, $a^2 = \frac{9}{4}b^2 = \frac{9}{4} \cdot 12 = 27$ and $c^2 = \frac{1}{4}b^2 = \frac{1}{4} \cdot 12 = 3$, i.e., $a = \sqrt{27} = 3\sqrt{3}$ and $c = \sqrt{3}$. The Lagrange multiplier is $\lambda = -36\sqrt{3}\pi$, while the minimal volume $24\sqrt{3}\pi$ is obtained for the ellipsoid

$$\frac{x^2}{27} + \frac{y^2}{12} + \frac{z^2}{3} = 1.$$

Alternative Solution: There are solutions which avoid Lagrange multipliers. One of them uses the famous "arithmetic mean – geometric mean inequality" (AM-GM inequality). For our problem, we only need the case for $n = 3$:

$$\sqrt[3]{x_1 x_2 x_3} \leq \frac{x_1 + x_2 + x_3}{3} \text{ for any } x_1, x_2, x_3 \geq 0.$$

Furthermore, equality is obtained if and only if the variables are equal to each other, i.e., $x_1 = x_2 = x_3$.

Going back to our bonus Midterm Problem #6, we want to find the minimum of $f(a, b, c) = \frac{4\pi}{3} abc$, given the constraint $g(a, b, c) = \frac{9}{a^2} + \frac{4}{b^2} + \frac{1}{c^2} = 1$, where $a, b, c > 0$. We have to cleverly choose here our numbers x_1, x_2, x_3, to which to apply the AM-GM inequality. How about applying AM-GM to the function $g(a, b, c)$:

$$g(a, b, c) = 1 = \frac{9}{a^2} + \frac{4}{b^2} + \frac{1}{c^2} \Rightarrow \frac{1}{3} = \frac{\frac{9}{a^2} + \frac{4}{b^2} + \frac{1}{c^2}}{3} \overset{\text{AM-GM}}{\geq} \sqrt[3]{\frac{9}{a^2} \cdot \frac{4}{b^2} \cdot \frac{1}{c^2}} = \sqrt[3]{\left(\frac{6}{abc}\right)^2}$$

$$\Leftrightarrow \left(\frac{1}{3}\right)^3 \geq \left(\frac{6}{abc}\right)^2 \Leftrightarrow \sqrt{\frac{1}{3^3}} \geq \frac{6}{abc} \Leftrightarrow abc \geq 6\sqrt{3^3} \Leftrightarrow \frac{4\pi}{3} \cdot abc \geq \frac{4\pi}{3} \cdot 6\sqrt{3^3} = 24\sqrt{3}\pi.$$

We have proven that the volume $f(a, b, c) = \frac{4\pi}{3} abc \geq 24\sqrt{3}\pi$. Equality can be attained if and only if the AM-GM yields equality, and we know that this happens exactly when the three variables $x_1 = \frac{9}{a^2}$, $x_2 = \frac{4}{b^2}$ and $x_3 = \frac{1}{c^2}$ are equal, i.e., $\frac{9}{a^2} = \frac{4}{b^2} = \frac{1}{c^2}$. At this point, we revert back to our original solution:

$$1 = g(a, b, c) = \frac{9}{a^2} + \frac{4}{b^2} + \frac{1}{c^2} = \frac{4}{b^2} + \frac{4}{b^2} + \frac{4}{b^2} = \frac{12}{b^2} \Rightarrow b^2 = 12, a^2 = 27, c^2 = 3,$$

i.e., $b = 2\sqrt{3}$, $a = 3\sqrt{3}$, $c = \sqrt{3}$, and the ellipsoid with minimal volume passing through $(3, 2, 1)$ is $\frac{x^2}{27} + \frac{y^2}{12} + \frac{z^2}{3} = 1$. □

Although this solution avoided using Lagrange multipliers, it did use the AM-GM inequality, which is typically proven via Lagrange multipliers. (This is why, in the first place, the AM-GM inequality appears in the exercises in section §14.8!) Still, the above AM-GM solution gives us a more direct approach to solving our problem and provides an insight into why $\frac{x^2}{27} + \frac{y^2}{12} + \frac{z^2}{3} = 1$ is truly the optimal ellipsoid.

5 No Calculators during the Exam. Cheat Sheet and Studying for the Exam

No calculators are allowed on the exam. Anyone caught using a calculator will be disqualified from the exam.

For the exam, you are allowed to have a "cheat sheet" - *one page* of a regular 8.5×11 sheet. You can write whatever you wish there, under the following conditions:

- The whole cheat sheet must be **handwritten by your own hand**! No xeroxing, no copying, (and for that matter, no tearing pages from the textbook and pasting them onto your cheat sheet.) DSP students with special writing or related disability should consult with the instructor regarding their cheat sheets.
- You must submit your cheat sheet on Gradescope by 11AM **before** the exam.
- Any violation of these rules will disqualify your cheat sheet and may end in your own disqualification from the exam. I may decide to randomly check your cheat sheets, so let's play it fair and square. :)

- Don't be a **freakasaurus**! Start studying for the exam several days in advance, and prepare your cheat sheet at least 2 days in advance. This will give you enough time to become familiar with your cheat sheet and be able to use it more efficiently on the exam.
- **Do NOT overstudy on the day of the exam!! No sleeping the night before the exam due to cramming, or more than 3 hours of math study on the day of the exam is counterproductive! No kidding!**